

A NILPOTENCY CRITERION FOR FINITE GROUPS

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Abstract. Let G be a finite group. We give a criterion of nilpotency of G based on the existence of elements of certain order in each section of G .

1. Introduction

The problem of detecting structural properties of finite groups by looking at element orders has been considered in many recent papers (see e.g. [1] and [3–6]). In the current note, we identify a new property detecting nilpotency of a finite group G that uses the function

$$\varphi(G) = |\{a \in G \mid o(a) = \exp(G)\}|$$

introduced and studied in [9]. The proof that we present is founded on the structure of minimal non-nilpotent groups (also called *Schmidt groups*) given by [8].

It is well known that a finite nilpotent group G contains elements of order $\exp(G)$. Moreover, all sections of G have this property. Under the above notation, this can be written alternatively as

$$(1) \quad \varphi(S) \neq 0 \quad \text{for any section } S \text{ of } G.$$

Our main theorem shows that the converse is also true, that is we have the following nilpotency criterion.

THEOREM 1. *Let G be a finite group. Then G is nilpotent if and only if $\varphi(S) \neq 0$ for any section S of G .*

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Note that (1) implies

$$(2) \quad \varphi(S) \neq 0 \quad \text{for any subgroup } S \text{ of } G$$

and in particular

$$(3) \quad \varphi(G) \neq 0.$$

We observe that the condition (3) is not sufficient to guarantee the nilpotency of G , as shows the elementary example $G = \mathbb{Z}_6 \times S_3$; we can even construct a non-solvable group G for which $\varphi(G) \neq 0$, namely $G = \mathbb{Z}_n \times H$, where H is a simple group of exponent n . A similar thing can be said about the condition (2).

EXAMPLE. Let G be a nontrivial semidirect product of a normal subgroup isomorphic to

$$E(5^3) = \langle x, y \mid x^5 = y^5 = [x, y]^5 = 1, [x, y] \in Z(E(5^3)) \rangle$$

by a subgroup $\langle a \rangle$ of order 3 such that a commutes with $[x, y]$. Then G is a non-CLT group of order 375, more precisely it does not have subgroups of order 75. We infer that its subgroups are: G , all subgroups contained in the unique Sylow 5-subgroup, all Sylow 3-subgroups, and all cyclic subgroups of order 15. Clearly, G satisfies the condition (2), but it is not nilpotent.

Finally, we note that our criterion can be used to prove the non-nilpotency of a finite group by looking to its sections. In [9] we have determined several classes of groups G satisfying $\varphi(G) = 0$, such as dihedral groups D_{2n} with n odd, non-abelian P -groups of order $p^{n-1}q$ ($p > 2$, q primes, $q \mid p-1$), symmetric groups S_n with $n \geq 3$, and alternating groups A_n with $n \geq 4$. These examples together with Theorem 1 lead to the following corollary.

COROLLARY 2. *If a finite group G contains a section isomorphic to one of the above groups, then it is not nilpotent.*

2. Proof of Theorem 1

We will prove that a finite group all of whose sections S satisfy $\varphi(S) \neq 0$ is nilpotent. Assume that G is a counterexample of minimal order. Then G is a Schmidt group since all its proper subgroups satisfy the hypothesis. By [8] (see also [2,7]) it follows that G is a solvable group of order $p^m q^n$ (where p and q are different primes) with a unique Sylow p -subgroup P and a cyclic Sylow q -subgroup Q , and hence G is a semidirect product of P by Q . Moreover, we have:

- if $Q = \langle y \rangle$ then $y^q \in Z(G)$;

- $Z(G) = \Phi(G) = \Phi(P) \times \langle y^q \rangle$, $G' = P$, $P' = (G')' = \Phi(P)$;
- $|P/P'| = p^r$, where r is the order of p modulo q ;
- if P is abelian, then P is an elementary abelian p -group of order p^r and P is a minimal normal subgroup of G ;
- if P is non-abelian, then $Z(P) = P' = \Phi(P)$ and $|P/Z(P)| = p^r$.

We infer that $S = G/Z(G)$ is also a Schmidt group of order $p^r q$ which can be written as semidirect product of an elementary abelian p -group P_1 of order p^r by a cyclic group Q_1 of order q (note that S_3 and A_4 are examples of such groups). Clearly, we have $\exp(S) = pq$. On the other hand, it is easy to see that

$$L(S) = L(P_1) \cup \{Q_1^x \mid x \in S\} \cup \{S\}.$$

Thus, the section S does not have cyclic subgroups of order pq and consequently $\varphi(S) = 0$, a contradiction. This completes the proof. $\square$

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